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A Heat Equation in Which the Diffusion Coefficient Changes Sign

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1. INTRODUCTION

Of concern are problems of the form

$$\begin{aligned}
&(\mathrm{DE}) \quad \sigma\partial_t u = \partial_x^2 u - q\partial_x u - pu + f \quad \text{for} \quad (x,t) \in (a,b)\times(0,T)\\
&\qquad\quad \text{where } -\infty < a < 0 < b < \infty,\ T>0,\ \partial_t = \partial/\partial t,\ \partial_x = \partial/\partial x,\\
&(\mathrm{BC}) \quad u(a,t) = u(b,t) = 0 \quad \text{for} \quad t\in(0,T), \\
&(\mathrm{IC}) \quad u(x,0) = u_0(x) \quad \text{for} \quad x \in (0,b)\\
&(\mathrm{TC}) \quad u(x,T) = u_T(x) \quad \text{for} \quad x\in(a,0);
\end{aligned} \tag{1.1}$$

here σ, p, q, f, u_0, u_T are given, $\sigma(x,t)$ is positive or negative accordingly as x is, and the compatibility conditions $u_0(b) = u_T(a) = 0$ hold. Problems of the type $\sigma\partial_t u = \partial_x^2 u$ with σ taking on both positive and negative values appear to have been considered first in 1913–1914 by Gevrey in [4] who specifically mentioned the case of $\sigma(x,t) = x^m$ with m an odd integer. Much later, in 1968, a detailed treatment of the case $\sigma(x,t) = x$ was worked out by Baouendi and Grisvard in [2]. A similar treatment in a context in which ∂_x^2 is replaced by a suitable nonlinear differential operator may be found in Lions' book [7, pp. 337–343]. Much of the present paper may be looked upon as a direct generalization of the ideas and methods of [2].

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Problems of the sort

$$\sigma(x, t, u, \partial_x u)\, \partial_t u = \partial_x^2 u,$$
$$\sigma(x, t, u, \partial_x u)\, \partial_t^2 u = \partial_x^2 u,$$

with σ taking on both positive and negative values arise in boundary layer problems in fluid dynamics (cf. [11, 12] and the references contained therein). A natural way to attack these nonlinear equations is by the method of successive approximations. (For definiteness we consider the parabolic case and omit the hyperbolic case.) If $u^{(1)}$ is an initial guess at a solution, then for $n \geqslant 2$ the nth approximate solution is obtained by solving the equation

$$\sigma(x, t, u^{(n-1)}(x, t), \partial_x u^{(n-1)}(x, t))\, \partial_t u^{(n)} = \partial_x^2 u^{(n)}$$

together with appropriate side conditions. One then hopes that the sequence $\{u^{(n)}\}$ converges to a solution of the original equation as $n \to \infty$. If the coefficient $\sigma(x, t, u, \partial_x u)$ genuinely depends on at least one of u, $\partial_x u$, then $\sigma(x, t, u^{(n)}, \partial_x u^{(n)}) = \sigma_n(x, t)$ depends on both x and t, even if σ does not depend explicitly on either x or t. Thus we view (1.1) as a model linear equation for more complicated equations which arise in fluid dynamics, and we insist that σ depend on both x and t.

In the next section we present an elementary treatment of the case of $\sigma(x, t) = x/|x| = \operatorname{Sgn} x$. For this diffusion coefficient the equation can be solved by separation of variables. We give a heuristic discussion of well-posed and ill-posed problems for this model equation (i.e., $(\operatorname{Sgn} x)\, \partial_t u = \partial_x^2 u$) and of the appropriateness of the initial and terminal conditions (IC), (TC), each being given on "half" the spatial interval (a, b). The existence and uniqueness theory for (1.1) is given in Sections 3 through 5. The remainder of the paper contains technical parts of the proofs.

2. A Concrete Example

We want to obtain solutions u for the problem

$$\begin{array}{lll} \text{(DE)} & (\operatorname{Sgn} x)\, \partial_t u = \partial_x^2 u & \text{for} \quad (x, t) \in [(-1, 0) \cup (0, 1)] \times (0, T), \\ \text{(BC)} & u(-1, t) = u(1, t) = 0 & \text{for} \quad t \in (0, T), \\ \text{(CC)} & u(0^+, t) = u(0^-, t) & \text{and} \quad \partial_x u(0^+, t) = \partial_x u(0^-, t) \\ & & \text{for} \quad t \in (0, T) \text{ (compatibility conditions)}, \end{array} \tag{2.1}$$

with initial and terminal conditions to be prescribed later. We apply separation of variables to obtain a solution u of the form

$$u(x,t)=\sum_{n=-\infty}^{\infty}(u_n(x,t)+v_n(x,t)) \tag{2.2}$$

where

$$\begin{aligned} u_n(x,t)&=e^{-\lambda_n^2 t}[f_1(\lambda_n)\,e^{i\lambda_n x}+e^{-i\lambda_n x}]\,B_n && \text{for} \quad x\in(0,1),\\ u_n(x,t)&=e^{-\lambda_n^2 t}[f_3(\lambda_n)\,e^{-\lambda_n x}+f_4(\lambda_n)\,e^{\lambda_n x}]\,B_n && \text{for} \quad x\in(-1,0), \end{aligned} \tag{2.3}$$

$$\begin{aligned} v_n(x,t)&=e^{\lambda_n^2 t}[f_3(\lambda_n)\,e^{\lambda_n x}+f_4(\lambda_n)\,e^{\lambda_n x}]\,b_n && \text{for} \quad x\in(0,1),\\ v_n(x,t)&=e^{\lambda_n^2 t}[f_1(\lambda_n)\,e^{i\lambda_n x}+e^{-i\lambda_n x}]\,b_n && \text{for} \quad x\in(-1,0). \end{aligned} \tag{2.4}$$

Here $f_2(\lambda_n)$, the coefficient of $e^{-i\lambda_n x}$, has been set equal to one as a normalization condition. A routine computation gives

$$f_1(\lambda)=-e^{-i2\lambda}, \qquad f_4(\lambda)=2^{-1}[(1-i)-(1+i)\,e^{-2i\lambda}],$$
$$f_3(\lambda)=2^{-1}[(1+i)-(1-i)\,e^{-2i\lambda}]=2^{-1}e^{-2i\lambda}[(1+i)\,e^{2i\lambda}-(1-i)].$$

The eigenvalue parameters $\{\lambda_n\}_{n=-\infty}^{\infty}$ are the roots of the equation

$$e^{-2\lambda}=\sec 2\lambda+\tan 2\lambda;$$

see Fig. 2.1. Thus $\{\lambda_n\}$ are the points of intersection of the curves $y=e^{-2\lambda}$ and $y=\sec 2\lambda+\tan 2\lambda$. We number them so that $\lambda_0=0$. Then (see Fig. 2.1),

$$\begin{aligned} &\text{for} \quad n>0, && 0<\lambda_n-\pi(n-1/4)\to 0 && \text{as} \quad n\to\infty,\\ &\text{for} \quad n<0, && 0<\pi(n+1/4)-\lambda_n\to 0 && \text{as} \quad n\to-\infty. \end{aligned}$$

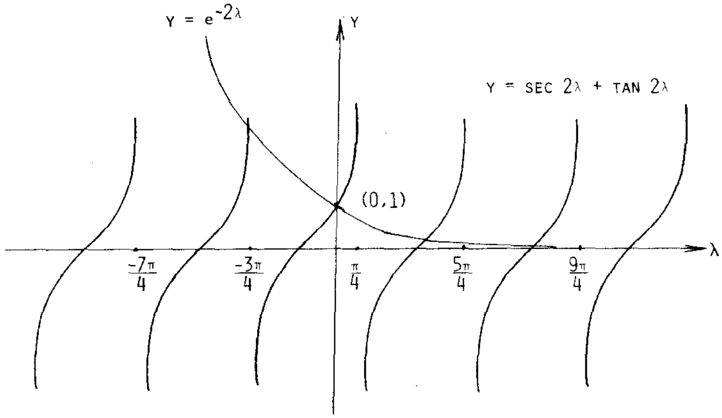

FIG. 2.1. Vertical asymptotes to the curve $y=\sec 2\lambda+\tan 2\lambda$ are the lines $\lambda=n\pi+\pi/4$ ($n=0,1,2,\ldots$). At the points $\lambda=n\pi-\pi/4$, $y=\sec 2\lambda+\tan 2\lambda$ satisfies $y=0$, $y'=1/2$.

Now let $\mathscr{X}$ be a Banach space of (possibly equivalence classes of) functions on $[-1, 1]$, containing $\{u \in C^1[-1, 1]: u(-1) = u(1) = 0\}$, and suppose $u_n(\cdot), v_n(\cdot) \in \mathscr{X}$, where

$$u_n(x, t) = u_n(x)\, e^{-\lambda_n^2 t}, \qquad v_n(x, t) = v_n(x)\, e^{\lambda_n^2 t}$$

together with (2.3), (2.4) define u_n, v_n. Since

$$\|v_n(\cdot, t)\|_{\mathscr{X}} / \|v_n(\cdot, 0)\|_{\mathscr{X}} = e^{\lambda_n^2 t} \to \infty$$

as $n \to \pm\infty$ it follows that the initial value problem for (2.1) is improperly posed in the sense that if $S(t)$ is the operator in $\mathscr{X}$ mapping the initial data $u_0 \in \mathscr{X}$ to the solution $u(\cdot, t) \in \mathscr{X}$ at time t, then

$$\|S(t)\| = \infty$$

for each $t > 0$. Similarly,

$$\lim_{n \to \pm\infty} \|u_n(\cdot, t)\|_{\mathscr{X}} / \|u_n(\cdot, 0)\|_{\mathscr{X}} = \infty$$

for each $t < 0$ implies that the terminal problem ((2.1) with

$$u(\cdot, T) = u_T(\cdot) \in \mathscr{X}$$

given) is improperly posed in $\mathscr{X}$.

We now specify initial and terminal data for (2.1) as

(IC) $\quad u(x, 0) = u_0(x)$ for $0 < x < 1$,

(TC) $\quad u(x, T) = u_T(x)$ for $-1 < x < 0$

(cf. (1.1).) Thus we specify initial data (resp. terminal data) on the right half (resp. left half) of the spatial interval where the problem is forward parabolic (resp. backward parabolic). See Fig. 2.2; the data is specified on the boldface lines.

Because of the infinite speed of propagation associated with the heat equation it is not at all clear intuitively (to us) if this problem (i.e., (1.1), or (2.1) together with (IC), (TC)) is well-posed. From an evolution equation point of view, well-posedness implies

$$\sup\{N(\text{solution at time } t)/N(\text{data})\} < \infty$$

where N represents a suitable norm. (Think of N as $\|\cdot\|_{\mathscr{X}}$.) If we take

$$N(\text{solution at } t)^2 = \int_{-1}^{1} |u(x, t)|^2\, dx,$$

$$N(\text{data})^2 = \int_{-1}^{0} |u_T(x)|^2\, dx + \int_0^1 |u_0(x)|^2\, dx,$$

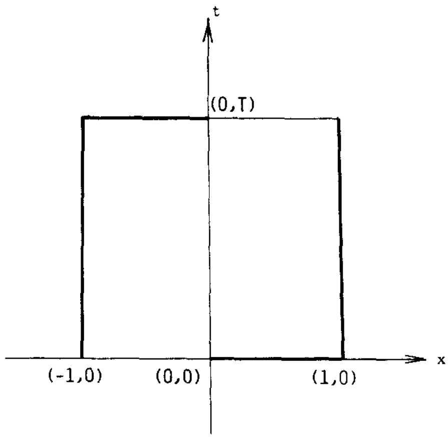

FIG. 2.2. Data given on boldface lines.

then by what was shown above

$$\sup\{N(\text{solution at } t)/N(\text{data}): u_0 \in C^\infty[0, 1],$$
$$u_0(1) = 0, u_T \in C^\infty[-1, 0], u_T(-1) = 0\} = \infty$$

for each $t \in (0, T)$. This suggests that (2.1), (IC), and (TC) (or (1.1)) are not well-posed from an evolution equation point of view. However, when looked upon as a "stationary problem" in a space of functions of two variables $(x, t) \in (-1, 1) \times (0, T)$ this problem becomes well-posed as will become clear in the sequel.

Alternatively, we can think of $\sigma\partial_t u = \partial_x^2 u$ as being a weakly elliptic problem in the space–time region Ω with the value of u being specified on part but not all of the boundary $\partial\Omega$. There is a substantial literature on "degenerate elliptic–parabolic" equations from this point of view; sample papers include [5, 9].

3. EXISTENCE OF A WEAK SOLUTION

We start by setting down the notation. $\mathbb{R}$ (resp. $\mathbb{C}$) denotes the real (resp. complex) numbers; $\mathbb{R}_+ = (0, \infty)$, $\bar{\mathbb{R}}_+ = [0, \infty)$. Points in $\mathbb{R} \times \mathbb{R}$ will generally be denoted by (x, t) with the spatial variable coming first and the time variable second. If S is an interval in $\mathbb{R}$ or a rectangle in $\mathbb{R}^2$ and if $\mathscr{X}$ is a Hilbert space, then $H^n(S; \mathscr{X})$, $H_0^n(S; \mathscr{X})$ will denote the usual Sobolev spaces, $n = 0, 1, 2$. Note that all derivatives are taken in the sense of distributions and $H^0 = H_0^0 = L^2$. $\mathscr{X}$ will usually be suppressed when $\mathscr{X} = \mathbb{C}$. $\mathscr{X}^*$ will denote the anti-dual space of $\mathscr{X}$, and $L^2(S) = L^2(S; \mathbb{C})$ will be identified with its anti-dual. If S is open, $\mathscr{D}(S)$ will be the Schwartz space of C^∞ functions with compact support in S; its anti-dual, the space of distributions on S, will be denoted by $\mathscr{D}'(S)$ (rather than $\mathscr{D}^*(S)$) as is customary. $C^n(\bar{\Omega})$ will denote the n times continuously differentiable functions on $\bar{\Omega}$. $C_0^n(\Omega)$ consists of those functions in $C^n(\bar{\Omega})$ which vanish off a compact subset of Ω. $C_0^\infty(\Omega)$ is $\mathscr{D}(\Omega)$. (See [3, 8, 10].)

Our basic domain will be $\Omega = (a, b) \times (0, T)$ where $-\infty < a < 0 < b < \infty$, $T > 0$. Note that $\bar{\Omega} = [a, b] \times [0, T]$. The (distributional) gradient of $u: \Omega \to \mathbb{C}$ will be denoted by $\nabla u = (\partial_x u, \partial_t u)$. Let I be the interval (a, b) and let $L = (a, 0)$, $R = (0, b)$ be its left and right "halves."

We let $\mathscr{H} = L^2(I)$, $\mathscr{H}^1 = H^1(I)$, $\mathscr{H}_0^1 = H_0^1(I)$, and $\mathscr{H}^{-1} = H^{-1}(I)$. These are all Hilbert spaces under the respective norms

$$\|u\|_0 = \left\{\int_a^b |u(x)|^2\, dx\right\}^{1/2},$$

$$\|u\|_1 = \left\{\int_a^b (|u(x)|^2 + |\partial_x u(x)|^2)\, dx\right\}^{1/2},$$

$$|||u|||_1 = \left\{\int_a^b |\partial_x u(x)|^2\, dx\right\}^{1/2},$$

$$\|u\|_{-1} = \sup\{|\langle u, \phi\rangle| : \phi \in \mathscr{H}_0^1, |||\phi|||_1 \leqslant 1\},$$

where the notation $\langle u, \phi\rangle$ has an obvious meaning. On $\mathscr{H}_0^1$ the norms $\|\cdot\|_1$ and $|||\cdot|||_1$ are equivalent because of the well-known inequality

$$\int_a^b |w(x)|^2\, dx \leqslant 2^{-1}(b-a)^2 \int_a^b |\partial_x w(x)|^2\, dx$$

for all $w \in \mathscr{H}_0^1$; this inequality will be used repeatedly in the sequel. The symbol $|\cdot|$ will be reserved to denote absolute value.

Let $F = L^2((0, T); \mathscr{H}_0^1)$, so that $F^* = L^2((0, T); \mathscr{H}^{-1})$. The symbol $\langle\cdot,\cdot\rangle$, without a subscript attached, will denote the conjugate duality between $\mathscr{D}'(I)$ and $\mathscr{D}(I)$, or between $H^{-1}(\mathbb{R})$ and $H^1(\mathbb{R})$, or between $H^{-1}(\mathbb{R}_+)$ and $H_0^1(\mathbb{R}_+)$, etc. With the single brackets $\langle\cdot,\cdot\rangle$ only one variable (e.g., x) will be involved. For functions of two variables we will use double angular brackets; thus $\langle\langle\cdot,\cdot\rangle\rangle$ will denote the conjugate duality between F' and F, or between $\mathscr{D}'(\Omega)$ and $\mathscr{D}(\Omega)$, or between $L^2(\mathbb{R}_+; H^{-1}(\mathbb{R}_+))$ and $L^2(\mathbb{R}_+; H_0^1(\mathbb{R}_+))$, etc. The context should make clear what is meant. When this is not the case we shall use appropriate subscripts.

Our minimal assumptions on the coefficients p, q, and σ are as follows.

HYPOTHESIS 1. (i) $p, q, \partial_x q \in L^\infty(\Omega)$.

(ii) $\sigma, \partial_t \sigma \in L^\infty(\Omega)$; $\sigma(x, 0) \geqslant 0$ in R; $\sigma(x, T) \leqslant 0$ in L.

(iii) $\partial_t \sigma(x, t) \leqslant c_1$ a.e. in Ω where c_1 is a constant and either

(iiia) q is real, $c_2 = \min\{0, \operatorname{ess\,inf}(p + \bar{p} - \partial_x q)\}$ and $c_1 < 4(b-a)^{-2} + c_2$, or

(iiib) $c_1 < 4(b-a)^{-2} - \sqrt{2}(b-a)^{-1} \|q\|_\infty + \min\{0, \operatorname{ess\,inf}(p + \bar{p})\}$.

In the simple case when $p \equiv q \equiv 0$, condition (iii) reduces to

$$\partial_t \sigma(x, t) \leqslant c_1 < 4(b-a)^{-2} \qquad \text{a.e.} \quad \text{in } \Omega.$$

Note that all derivatives are in the sense of distributions, so in (i), $q, \partial_x q \in L^\infty(\Omega)$ is equivalent to $q \in W^{1,\infty}(\Omega)$.

Recall that F is a Hilbert space under the norm

$$\|u\|_F = \left\{\int_0^T \int_a^b |\partial_x u(x, t)|^2 \, dx \, dt\right\}^{1/2} = \left\{\int_\Omega |\partial_x u|^2\right\}^{1/2};$$

we will not hesitate to omit arguments of integrands and differentials for notational simplicity. Define

$$\Phi = \{\phi \in F \cap C^1(\bar{\Omega}) \colon \phi(x, 0) = 0 \text{ in } L, \phi(x, T) = 0 \text{ in } R\}. \tag{3.1}$$

Φ is an inner product space whose norm is

$$\|\phi\|_\Phi = \left\{\|\phi\|_F^2 + \int_0^b \sigma(x, 0) \, |\phi(x, 0)|^2 \, dx - \int_0^a \sigma(x, T) \, |\phi(x, T)|^2 \, dx\right\}^{1/2}.$$

Let u_0, u_T be given and suppose

$$\begin{aligned} |\sigma(\cdot, 0)|^{1/2} \, u_0(\cdot) &\in L^2(R), \\ |\sigma(\cdot, T)|^{1/2} \, u_T(\cdot) &\in L^2(L), \end{aligned} \tag{3.2}$$

together with $f \in F^*$. Then we have the following result asserting the existence of a weak solution.

3.1. WEAK EXISTENCE THEOREM. *Assume Hypothesis* 1 *and* (3.2). *Then for each $f \in F^*$ there exists a $u \in F$ such that for all $\phi \in \Phi$,*

$$\begin{aligned} &-\int_\Omega u \partial_t(\sigma \bar{\phi}) + \int_\Omega (\partial_x u)(\partial_x \bar{\phi}) - \int_\Omega u \partial_x(q \bar{\phi}) + \int_\Omega p u \bar{\phi} \\ &\quad = \langle\langle f, \phi \rangle\rangle + \int_0^b \sigma(x, 0) \, u_0(x) \, \overline{\phi(x, 0)} \, dx - \int_a^0 \sigma(x, T) \, u_T(x) \, \overline{\phi(x, T)} \, dx. \end{aligned} \tag{3.3}$$

Proof. Note first that if u is a classical solution of (1.1), then integration by parts shows that u satisfies (3.3). This is the sense in which u is a weak solution.

The proof will depend on the Lions' Projection Theorem [6, p. 37]. Define $M: \Phi \to \mathbb{C}$ and $E: F \times \Phi \to \mathbb{C}$ by

$$M(\phi) = \text{right hand side of (3.3)},$$

$$E(u, \phi) = E_1(u, \phi) + E_2(u, \phi),$$

where

$$E_1(u, \phi) = -\int_\Omega u\partial_t(\sigma\bar{\phi}) + \int_\Omega (\partial_x u)(\partial_x \bar{\phi}),$$

$$E_2(u, \phi) = -\int_\Omega u\partial_x(q\bar{\phi}) + \int_\Omega pu\bar{\phi}.$$

Clearly M is continuous and anti-linear, while E is sesquilinear and continuous in the first variable. Since the inclusion map $\Phi \hookrightarrow F$ is continuous, we deduce the existence of a $u \in F$ satisfying

$$E(u, \phi) = M(\phi) \qquad \text{for all } \phi \in \Phi,$$

thereby proving that (3.3) holds, provided we can find an $\varepsilon > 0$ such that

$$|E(\phi, \phi)| \geqslant \varepsilon \, \|\phi\|_\Phi^2 \tag{3.4}$$

holds for all $\phi \in \Phi$ (cf. [6, p. 37]).

A straightforward calculation shows that for all $\phi \in \Phi$,

$$\begin{aligned} 2\,\mathrm{Re}\, E_1(\phi, \phi) &= -\int_\Omega (\partial_t \sigma)\, |\phi|^2 + 2\, \|\phi\|_F^2 \\ &\quad - 2\int_a^b \sigma(x, T)\, |\phi(x, T)|^2\, dx + 2\int_a^b \sigma(x, 0)\, |\phi(x, 0)|^2\, dx \\ &\geqslant -c_1 \int_\Omega |\phi|^2 + 2\, \|\phi\|_\Phi^2 \geqslant [-2^{-1} c_1 (b-a)^2 + 2]\, \|\phi\|_\Phi^2; \end{aligned} \tag{3.5}$$

$$2\,\mathrm{Re}\, E_2(\phi, \phi) = 2\,\mathrm{Re} \int_\Omega \{(-\partial_x q + p)\, |\phi|^2 - q\phi\partial_x\bar{\phi}\}. \tag{3.6}$$

If q is real-valued,

$$2\,\mathrm{Re} \int_\Omega q\phi\partial_x\bar{\phi} = -\int_\Omega \partial_x q\, |\phi|^2,$$

whence

$$2\operatorname{Re} E_2(\phi,\phi)=\operatorname{Re}\int_\Omega(-\partial_x q+2p)\,|\phi|^2$$

$$\geqslant c_2\int_\Omega|\phi|^2\geqslant 2^{-1}c_2(b-a)^2\,\|\phi\|_\Phi^2 \tag{3.7}$$

if (iiia) holds. Combining (3.5) and (3.7) gives

$$2\operatorname{Re} E(\phi,\phi)\geqslant 2^{-1}(b-a)^2[c_2-c_1+4(b-a)^{-2}]\,\|\phi\|_\Phi^2.$$

If we choose

$$\varepsilon\in(0,4^{-1}(b-a)^2[4(b-a)^{-2}-c_1+c_2])$$

we arrive at (3.4), thus obtaining the existence of a solution u of (3.3).

Whether or not q is real, (3.6) yields

$$2\operatorname{Re} E_2(\phi,\phi)\geqslant -2\,\|q\|_\infty\left(\int_\Omega|\phi|^2\right)^{1/2}\left(\int_\Omega|\partial_x\phi|^2\right)^{1/2}+2\operatorname{Re}\int_\Omega p\,|\phi|^2$$

$$\geqslant -\sqrt{2}(b-a)\,\|q\|_\infty\,\|\phi\|_F^2+\int_\Omega(p+\bar{p})\,|\phi|^2,$$

and, as before, we arrive at (3.4) with a proper choice of $\varepsilon>0$. ∎

3.2. *Remark.* The solution u obtained above by the Lions' Projection Theorem is the image of a certain linear operator A on the unique $\xi_M\in\Phi$ defined by $M(\phi)=\langle\langle\xi_M,\phi\rangle\rangle_\Phi$ for all $\phi\in\Phi$. This operator A depends on the explicit form of E but not on that of M. The solution u thus obtained is unique in a certain sense, namely, the sequence of steps prescribed to construct the solution will always lead to the answer $A\xi_M=u$, allowing us to call such a solution a *canonical solution.* Note that the solution $u=A\xi_M$ has not yet been shown to be unique in F; this we intend to do eventually.

3.3. *Remark.* It follows from [6, p. 37] that the canonical solution u satisfies

$$\|u\|_F\leqslant\varepsilon^{-1}N(M)$$

where ε is given by (3.4) and

$$N(M)=\sup\{|M(\phi)|:\phi\in\Phi,\|\phi\|_\Phi\leqslant 1\}$$
$$\leqslant\|f\|_{F^*}+\|\,|\sigma(\cdot,0)|^{1/2}\,u_0\|_{L^2(R)}+\|\,|\sigma(\cdot,T)|^{1/2}\,u_T\|_{L^2(L)}.$$

As is shown in [6, pp. 41–42], we can deduce that if v is the canonical solution of the corresponding problem obtained by replacing f, u_0, u_T by g, v_0, v_T respectively, then

$$\|u-v\|_F \leqslant \varepsilon^{-1}\{\|f-g\|_{F^*} + \| \, |\sigma(\cdot,0)|^{1/2}(u_0-v_0)\|_{L^2(R)} + \| \, |\sigma(\cdot,T)|^{1/2}(u_T-v_T)\|_{L^2(L)}\}.$$

This stability criterion shows that the canonical solution u depends continuously on f, u_0, u_T, and so the problem is well-posed in a certain sense.

3.4. *Remark.* In Theorem 3.1 the solution u is in $F=L^2((0,T);\mathscr{H}_0^1)$, whence $u(\cdot,t)\in\mathscr{H}_0^1$ for a.a. t, so that the boundary conditions $u(a,t)=u(b,t)=0$ automatically hold in an almost everywhere sense.

3.5. COROLLARY. *The solution u obtained in Theorem* 3.1 *is a solution of*

$$\sigma\partial_t u = \partial_x^2 u - q\partial_x u - pu + f \tag{3.8}$$

in $\mathscr{D}'(\Omega)$ as well as in F^*. In particular, $\sigma\partial_t u\in F^*$.

3.6. *Remark.* The preceding corollary shows that the solution of the weak equation (3.3) is in the space

$$\mathscr{B} = \{u\in F: \sigma\partial_t u\in F^*\}. \tag{3.9}$$

$\mathscr{B}$ becomes an inner product space under the norm

$$\|u\|_{\mathscr{B}} = \{\|u\|_F^2 + \|\sigma\partial_t u\|_{F^*}^2\}^{1/2} \tag{3.10}$$

for $u\in\mathscr{B}$; this will be used later.

4. STRONG SOLUTIONS

4.1. STRONG UNIQUENESS THEOREM. *Assume Hypothesis* 1. *Then any solution of* (3.3) *satisfying $u\in F$ and $\partial_t u$, $\partial_x u$, $\partial_x^2 u\in L^2((0,T);\mathscr{H})$ is unique.*

Proof. Let u satisfy (3.3) with $f=u_0=u_T=0$, and with $u\in F$ and $\partial_t u$, $\partial_x u$, $\partial_x^2 u\in L^2((0,T);\mathscr{H})$. It is enough to show that $u=0$.

For this we take the inner product of $\sigma\partial_t u - \partial_x^2 u + q\partial_x u + pu = 0$ with u in $L^2((0,T);\mathscr{H})$ and integrate by parts. Because of our hypotheses we obtain, after some calculation,

$$\|u\|_F^2 + \int_\Omega [q(\partial_x u)\bar{u} + p|u|^2] \leqslant 4^{-1}c_1(b-a)^2\|u\|_F^2.$$

Condition (iii) of Hypothesis 1 then implies $\|u\|_F = 0$. ∎

Next we consider the following special case of (3.3).

$$\begin{array}{lll} \text{(DE)} & \sigma\partial_t u = \partial_x^2 u + f & \text{on } \Omega, \\ \text{(BC)} & u(a,t) = 0 = u(b,t) & \text{for } \quad t \in (0,T), \\ \text{(IC)} & u(x,0) = u_0(x) & \text{on } R \text{ where } |\sigma(\cdot,0)|^{1/2}\, u_0 \in L^2(R), \\ \text{(TC)} & u(x,T) = u_T(x) & \text{on } L \text{ where } |\sigma(\cdot,T)|^{1/2}\, u_T \in L^2(L). \end{array} \tag{4.1}$$

We obtain a strong solution of (4.1) (in the sense of Theorem 4.1) as follows.

4.2. THEOREM. *Let conditions* (i) *and* (ii) *of Hypothesis* 1 *hold and assume the following*:

(i) $\partial_t \sigma(x,t) - 2 \operatorname{ess\,inf}_\Omega \partial_t \sigma \leqslant \text{const} < 4(b-a)^{-2}$ *a.e. in* Ω;

(ii) $\sigma \in C^1(\bar{\Omega})$, $(\operatorname{Sgn} x)\, \sigma(x,t) > 0$ *for* $x \in I \backslash \{0\}$ *and* $t \in [0,T]$;

(iii) $u_0 \in H^2(R)$, $u_0(b) = 0$, $u_T \in H^2(L)$, $u_T(a) = 0$;

(iv) $f \in C^1(\bar{\Omega})$ *and* $\partial_t f \in F^*$;

(v) $|\sigma(\cdot,0)|^{-1/2}[f(\cdot,0) + \partial_x^2 u_0] \in L^2(R)$ *and*
$|\sigma(\cdot,T)|^{-1/2}[f(\cdot,T) + \partial_x^2 u_T] \in L^2(L)$.

Then (4.1) *has a strong solution.*

Proof. By formally looking at the weak solution u of (4.1) we obtain the following problem for $v = \partial_t u$.

$$\begin{array}{lll} \text{(DE)} & \sigma\partial_t v - \partial_x^2 v + (\partial_t \sigma)\, v = \partial_t f & \text{in } F^*, \\ \text{(BC)} & v(a,t) = 0 = v(b,t) & \text{for } \quad t \in [0,T), \\ \text{(IC)} & v(x,0) = v_0(x) \equiv \sigma(x,0)^{-1}(f(x,0) + \partial_x^2 u_0(x)) & \text{for } \quad x \in R, \\ \text{(TC)} & v(x,T) = v_T(x) \equiv \sigma(x,T)^{-1}(f(x,T) + \partial_x^2 u_T(x)) & \text{for } \quad x \in L. \end{array} \tag{4.2}$$

By Theorem 3.1, (4.2) has a weak solution v belonging to F. Thus $v \in L^2((0,T); \mathscr{H}_0^1)$ and $\sigma\partial_t v \in L^2((0,T); \mathscr{H}^{-1})$. It follows that

$$\sigma v \in L^2((0,T); \mathscr{H}_0^1), \qquad \partial_t(\sigma v) \in L^2((0,T); \mathscr{H}^{-1})$$

(since $\sigma \in C^1(\bar{\Omega})$). Thus $\sigma v \in C([0,T]; \mathscr{H})$ by [3, p. 176]. Thus $v = \sigma v/\sigma \in C([0,T]; \mathscr{H})$ because of condition (ii) and $v \in L^2([0,T]; \mathscr{H}_0^1)$. Choose a representative $v(\cdot,\cdot)$ of v mapping $\bar{\Omega}$ to $\mathbb{C}$ which is jointly measurable and such that $v(\cdot,t) \in \mathscr{H}_0^1$ for a.e. $t \in [0,T]$. Then for u_1 defined by

$$\begin{aligned} u_1(x,t) &= u_T(x) - \int_t^T v(x,\tau)\,d\tau \qquad \text{for} \quad x \in [a, 0), \\ &= u_0(x) + \int_0^t v(x,\tau)\,d\tau \qquad \text{for} \quad x \in (0, b], \end{aligned} \tag{4.3}$$

we have

$$\partial_t u_1(x,t) = v(x,t)$$

for a.e. $t \in [0, T]$ and for $x \notin N_t$ where N_t is a Lebesgue null set in I (containing 0). Thus $v = \partial_t u_1 \in F$. Clearly

$$\partial_x^j u_1(\cdot, t) = \partial_x^j u_0 + \int_0^t \partial_x^j v(\cdot, \tau)\,d\tau$$

for $j = 0, 1, 2$ in $\mathscr{D}'(R)$, and a similar result holds in $\mathscr{D}'(L)$. By (DE) of (4.2),

$$\begin{aligned} \partial_x^2 u_1(t) &= \partial_x^2 u_0 + \int_0^t [-\partial_t f(\tau) + \partial_t(\sigma v)(\tau)]\,d\tau \\ &= \partial_x^2 u_0 - f(t) + f(0) + (\sigma v)(t) - (\sigma v)(0) \\ &= (\sigma v)(t) - f(t) \qquad \text{in } \mathscr{D}'(R) \end{aligned}$$

by (IC) of (4.2). Similarly

$$\partial_x^2 u_1(t) = (\sigma v)(t) - f(t) \qquad \text{in } \mathscr{D}'(L).$$

Since σv and f are continuous functions of t from $[0, T]$ to $\mathscr{H}$ we conclude

$$t \mapsto \partial_x^2 u_1(t) = (\sigma v)(t) - f(t) \in C([0, T]; \mathscr{H}).$$

Consequently

$$(\sigma \partial_t u_1)(x,t) = f(x,t) + \partial_x^2 u_1(x,t)$$

for a.e. $(x, t) \in \bar{\Omega}$. Thus

$$\sigma \partial_t u_1 - \partial_x^2 u_1 = f$$

in $L^2((0, T); \mathscr{H})$. One can check directly from (4.3) that the appropriate side conditions are satisfied, and so it follows that u_1 is a strong solution of (4.1). ∎

5. Weak Uniqueness

The question of uniqueness of weak solutions is much harder. We begin by imposing a number of conditions on the diffusion coefficient σ. We list them all here for ready reference. These hypotheses are not as bad as they appear at first glance; see Remark 5.2.

Hypothesis 2. $\sigma: \bar{\Omega} \to \mathbb{R}$ satisfies: given b_1 with $a < -b_1 < 0 < b_1 < b$, there is an $m > 0$ such that $(\operatorname{Sgn} x)\,\sigma(x,t) \geqslant m$ provided $(x,t) \in \bar{\Omega}$ and $|x| \geqslant b_1$.

Thus σ strictly changes sign at $x = 0$. This strengthens Hypothesis 1(ii).

Hypothesis 3. σ extends to a function $\sigma_0: \mathbb{R}^2 \to \mathbb{R}$ with restrictions $\sigma_1: \mathbb{R} \times \bar{\mathbb{R}}_+ \to \mathbb{R}$, $\sigma_+: \bar{\mathbb{R}}_+ \times \bar{\mathbb{R}}_+ \to \mathbb{R}$ such that σ_0 is the even extension (in t) of σ_1, and $(\operatorname{Sgn} x)\,\sigma_1(x,t) \geqslant 0$. Moreover σ_1, $\partial_t |\sigma_1|$, $\partial_x |\sigma_1|$, $\partial_x \partial_t |\sigma_1| \in L^\infty(\mathbb{R} \times \mathbb{R}_+)$.

Hypothesis 4. There exist functions $\alpha_i: (-\infty, 0] \to \mathbb{R}$ and γ_i, g_i: $(-\infty, 0] \to \bar{\mathbb{R}}_+$ $(i = 1, 2)$ satisfying:

(i) $\alpha_i \in L^\infty$, $\alpha_i' \in L^\infty_{\text{loc}}(-\infty, 0]$, $\alpha_1(0) + \alpha_2(0) = 1$;

(ii) $g_i \in C^1(-\infty, 0]$, g_i is surjective, $g_1(0) = 0$, $\lim_{x \to -\infty} g_i(x) = \infty$, $g_i'(x) \leqslant -\mu < 0$ for some constant μ and all $x \leqslant 0$;

(iii) $\alpha_1(0)/g_1'(0) + \alpha_2(0)/g_2'(0) = -1$, $|\alpha_i'(x)|^2 \leqslant -\gamma_i(x)\, g_i'(x)$ a.e.;

(iv) $\gamma_i(g_i^{-1})$, $|\alpha_i(g_i^{-1})|\,|g_i'(g_i^{-1})| \in L^\infty$,

$$(x,t) \mapsto \delta_i(x,t) \equiv \frac{\alpha_i(g_i^{-1}(x))\, \sigma_1(g_i^{-1}(x), t)}{\sigma_1(x,t)} \in L^\infty(\mathbb{R}_+ \times \mathbb{R}_+);$$

(v) $\beta_i(x,t) \equiv \delta_i(x,t)/g_i'(g_i^{-1}(x))$ satisfies $D\beta_i \in L^\infty_{\text{loc}}$ for $D =$ identity, ∂_x, ∂_t, $\partial_x \partial_t$;

(vi) $\beta_i(g_i(x), t)^2\, g_i^{-1}(g_i(x))$, $\quad |\partial_x \beta_i(g_i(x), t)|^2\, g_i'(x) \in L^\infty((-\infty, 0) \times \mathbb{R}_+)$.

Hypothesis 5. σ_0 can be factored as $\sigma_0(x,t) = \xi(x)\,\eta(x,t)$ where $\eta \in W^{1,\infty}(\mathbb{R}^2) \cap W^{2,\infty}_{\text{loc}}(\mathbb{R}^2)$, $\eta^{-1} \in W^{2,\infty}_{\text{loc}}(\mathbb{R}^2)$, and η^{-1}, $\partial_x \eta^{-1} \in L^\infty(J \times \mathbb{R})$ for each bounded interval J, and $\xi \in W^{1,\infty}_{\text{loc}}(\mathbb{R})$.

5.1. Example. Let $\sigma(x,t) = x^m$ with m an odd integer. Let

$$\begin{aligned} \sigma_0(x) &= x^m && \text{if } |x| \leqslant a_1 = \max\{-a, b\} \\ &= (\operatorname{Sgn} x)\, a_1^m && \text{if } |x| > a_1. \end{aligned}$$

Then it is easy to check that Hypotheses 1–5 are all satisfied if q is real and ess $\inf\{2\,\mathrm{Re}(p) - \partial_x q\} \geqslant -4(b-a)^2$, and if we take $c_1 = 0$, $\alpha_1 \equiv 4$, $\alpha_2 \equiv -3$, $g_1(x) \equiv -2x$, $g_2(x) \equiv -3x$, and $\gamma_1(x) \equiv \gamma_2(x) \equiv 0$.

5.2. *Remark.* Example 5.1 is much more general than it appears. Suppose η_0 is a C^2 function from $\bar{\Omega}$ to $(0, \infty)$ and that $\xi_0 \colon \bar{I} \to \mathbb{R}$ is C^2 and strictly increasing with $\xi_0(0) = 0$. Let $\eta \in C^2(\mathbb{R}^2)$ extend η_0 and have compact range in $(0, \infty)$ and be an even function of t. Let $\xi \in C^2(\mathbb{R})$ be a bounded nondecreasing extension of ξ_0. Then Hypotheses 1–5 hold with $\sigma(x, t) = \xi(x)\,\eta(x, t)$ if the assumptions in the last sentence of Example 5.1 hold.

Thus the many conditions listed in Hypotheses 1–5 are not as bad as they seem, as they are easy to verify in a number of nontrivial situations. The reason for the complicated statements of the hypotheses is generality; we have nonlinear applications in mind.

5.3. THEOREM (Green's Formula). *Assume Hypotheses* 2–5.

(a) *If* $u \in \mathscr{B}$ *(see* (3.9)), *then* $\sigma(\cdot, 0)\,u(\cdot, 0)$ *and* $\sigma(\cdot, T)\,u(\cdot, T)$ *are well-defined as elements of* $\mathscr{H}$.

(b) *For all* $u, v \in \mathscr{B}$,

$$\langle\!\langle \partial_t(\sigma u), v \rangle\!\rangle + \overline{\langle\!\langle \sigma \partial_t v, u \rangle\!\rangle}$$
$$= \int_a^b \sigma(x, T)\,u(x, T)\,\overline{v(x, T)}\,dx - \int_a^b \sigma(x, 0)\,u(x, 0)\,\overline{v(x, 0)}\,dx. \tag{5.1}$$

Here the double angular brackets denote the F^*–F conjugate duality. This theorem will be proved in Sections 6–8. In this section we assume its validity and determine the sense in which the weak solution of Theorem 3.1 satisfies (IC) and (TC).

5.4. THEOREM. *Assume Hypotheses* 1–5 *and* (3.2). *If* $u \in F$ *is the solution obtained in Theorem* 3.1, *then the traces* $u(\cdot, 0)$, $u(\cdot, T)$ *of this solution satisfy the following conditions. For all* $g \in C_0^1(R)$ *and all* $h \in C_0^1(L)$,

$$\int_0^b \sigma(x, 0)\,u(x, 0)\,\overline{g(x)}\,dx = \int_0^b \sigma(x, 0)\,u_0(x)\,\overline{g(x)}\,dx, \tag{5.2}$$

$$\int_a^0 \sigma(x, T)\,u(x, T)\,\overline{h(x)}\,dx = \int_a^0 \sigma(x, T)\,u_T(x)\,\overline{h(x)}\,dx. \tag{5.3}$$

Proof. We will first prove that for all $\phi \in \Phi$ (see (3.1)),

$$\int_0^b \sigma(x,0)\, u(x,0)\, \bar{\phi}(x,0)\, dx - \int_a^0 \sigma(x,T)\, u(x,T)\, \bar{\phi}(x,T)\, dx$$
$$= \int_0^b \sigma(x,0)\, u_0(x)\, \bar{\phi}(x,0)\, dx - \int_a^0 \sigma(x,T)\, u_T(x)\, \bar{\phi}(x,T)\, dx. \quad (5.4)$$

Since $\partial_x^2 u \in F^*$, we obtain from (3.3) and (3.8) that for all $\phi \in \Phi$,

$$-\iint_\Omega u\partial_t(\sigma\bar{\phi})\, dx\, dt - \langle\langle \partial_x^2 u + q\partial_x u + pu, \phi\rangle\rangle$$
$$= \langle\langle \sigma\partial_t u - \partial_x^2 u + q\partial_x u + pu, \phi\rangle\rangle$$
$$+ \int_0^b \sigma(x,0)\, u_0(x)\, \bar{\phi}(x,0)\, dx - \int_a^0 \sigma(x,T)\, u_T(x)\, \bar{\phi}(x,T)\, dx. \quad (5.5)$$

(This is valid for $\phi \in \mathscr{D}(\bar{\Omega})$ and follows for $\phi \in F$ by a passage to the limit.)

Substitute

$$\iint_\Omega u\partial_t(\sigma\bar{\phi}) = \langle\langle (\partial_t\sigma)\, u, \phi\rangle\rangle + \overline{\langle\langle \sigma\partial_t\phi, u\rangle\rangle}$$

into (5.5) and combine the resulting equation with the one obtained by replacing v with ϕ in (5.1). (Note that $\Phi \subset \mathscr{B}$.) This gives (5.4).

Now let $g \in C_0^1(R)$. Extend g to I by letting it vanish outside R. Now define $\phi: \bar{\Omega} \to \mathbb{C}$ by $\phi(x,t) = [(T-t)/T]\, g(x)$. Then $\phi \in \Phi$, and plugging this ϕ into (5.4) gives (5.2). Equation (5.3) is obtained similarly. ∎

Thus the traces $u(x,0)$, $u(x,T)$ of the weak solution u satisfy the equation

$$\sigma(\cdot,0)\, u(\cdot,0) = \sigma(\cdot,0)\, u_0 \qquad \text{in } \mathscr{D}'(R),$$
$$\sigma(\cdot,T)\, u(\cdot,T) = \sigma(\cdot,T)\, u_T \qquad \text{in } \mathscr{D}'(L).$$

By Hypothesis 2,

$$u(\cdot,0) = u_0 \text{ in } \mathscr{D}'(R), \qquad u(\cdot,T) = u_T \text{ in } \mathscr{D}'(L). \quad (5.6)$$

This makes precise the sense in which the weak solution u satisfies the initial and terminal conditions.

5.5. Uniqueness Theorem for Weak Solutions. *Let Hypotheses* 1–5 *and* (3.2) *hold. For each $f \in F^*$ there is only one weak solution u in F of* (1.1) (*in the sense of* (3.3)).

Proof. It is enough to show that a solution $u \in F$ of (3.3) is zero provided $f = u_0 = u_T = 0$.

By Theorem 5.4 (see (5.6)), $u(x, 0) = 0$ for a.e. $x \in R$ and $u(x, T) = 0$ for a.e. $x \in L$.

Taking $v = u$ in (5.1) gives

$$2 \operatorname{Re} \langle\!\langle \sigma \partial_t u, u \rangle\!\rangle + \langle\!\langle (\partial_t \sigma) u, u \rangle\!\rangle = 0. \tag{5.7}$$

Next take a sequence $\{\phi_n\} \subset \mathscr{D}(\Omega)$ converging to u in F. Using (3.8) we obtain

$$\begin{aligned} -\langle\!\langle \sigma \partial_t u, u \rangle\!\rangle &= -\lim_{n \to \infty} \langle\!\langle \sigma \partial_t u, \phi_n \rangle\!\rangle \\ &= \lim_{n \to \infty} \iint_\Omega [(\partial_x u)(\partial_x \bar{\phi}_n) + q(\partial_x u)\bar{\phi}_n + p u_n \bar{\phi}_n] \\ &= \iint_\Omega [|\partial_x u|^2 + q(\partial_x u)\bar{u} + p|u|^2], \end{aligned} \tag{5.8}$$

and so

$$\begin{aligned} 2\|u\|_F^2 &+ 2 \operatorname{Re} \iint_\Omega q(\partial_x u)\bar{u} + 2 \operatorname{Re} \iint_\Omega p|u|^2 \\ &= -2 \operatorname{Re} \langle\!\langle \sigma \partial_t u, u \rangle\!\rangle \qquad \text{by (5.8)} \\ &= \langle\!\langle (\partial_t \sigma) u, u \rangle\!\rangle \qquad \text{by (5.7)} \\ &= \iint_\Omega (\partial_t \sigma)|u|^2 \leqslant c_1 \iint_\Omega |u|^2 \leqslant c_1 (b-a)^2 \|u\|_F^2/2, \end{aligned}$$

which contradicts each of Conditions (iiia), (iiib) of Hypothesis 1 unless $\|u\|_F = 0$. ∎

6. Proof of Green's Formula

Recall the definition of the Hilbert space $\mathscr{B}$ (see Remark 3.6).

6.1. Lemma. *Assume Hypothesis* 5. *Then*

$$\mathscr{B} = \{u \in F \colon \xi \partial_t u \in F^*\}. \tag{6.1}$$

Proof. Recall the factorization $\sigma(x, t) = \xi(x)\,\eta(x, t)$ for $(x, t) \in \Omega$. Let $u \in \mathscr{B}$. By Hypothesis 5, $\xi u \in F$ and $\xi \partial_t u = \partial_t(\xi u) \in H^{-1}((0, T); \mathscr{H}^{-1})$. For $\phi \in \mathscr{D}(\Omega)$ we *claim* that

$$\langle\!\langle \xi \partial_t u, \phi \rangle\!\rangle = \langle\!\langle \sigma \partial_t u, \eta^{-1} \phi \rangle\!\rangle \tag{6.2}$$

(the first pairing being that of $\mathscr{D}'(\Omega)$–$\mathscr{D}(\Omega)$ and the second being that of F^*–F). To see why this is so, let $\{\phi_n\} \subset \mathscr{D}(\Omega)$ converge to $\eta^{-1}\phi$ in $H_0^2(\Omega)$ and so in F. Then $\eta\phi_n$ converges to ϕ in $H_0^2(\Omega)$. Thus

$$\begin{aligned}
\langle\langle \sigma\partial_t u, \phi_n\rangle\rangle &= \langle\langle \partial_t(\sigma u), \phi_n\rangle\rangle - \langle\langle (\partial_t\sigma)\, u, \phi_n\rangle\rangle \\
&= -\langle\langle \sigma u, \partial_t\phi_n\rangle\rangle - \langle\langle (\partial_t\sigma)\, u, \phi_n\rangle\rangle \\
&= -\iint_\Omega [\sigma u \partial_t(\bar{\phi}_n) + (\partial_t\sigma)\, u\bar{\phi}_n] = -\iint_\Omega \xi u \partial_t(\eta\bar{\phi}_n) \\
&= -\lim_{m\to\infty} \iint_\Omega u_m \partial_t(\eta\bar{\phi}_n) \qquad \text{where} \quad \mathscr{D}(\Omega) \ni u_m \to \xi u \text{ in } F \\
&= -\lim_{m\to\infty} \iint_\Omega \partial_t(u_m\eta\bar{\phi}_n) + \lim_{m\to\infty} \iint_\Omega (\partial_t u_m)\, \eta\bar{\phi}_n \\
&= \lim_{m\to\infty} \langle\langle \partial_t u_m, \eta\varphi_n\rangle\rangle = \langle\langle \xi\partial_t u, \eta\phi_n\rangle\rangle,
\end{aligned}$$

where the last two pairings are $H^{-2}(\Omega)$–$H_0^2(\Omega)$. Letting $n \to \infty$ gives (6.2).

It follows from (6.2) that

$$|\langle\langle \xi\partial_t u, \phi\rangle\rangle| \leqslant \text{Const}\, \|\phi\|_F,$$

and so, by the density of $\mathscr{D}(\Omega)$ in F, we conclude that $\xi\partial_t u \in F^*$.

Conversely, let $u \in F$ with $\xi u \in F^*$. Then for all $\phi \in \mathscr{D}(\Omega)$ we *claim* that

$$\langle\langle \sigma\partial_t u, \phi\rangle\rangle = \langle\langle \xi\partial_t u, \eta\phi\rangle\rangle. \tag{6.3}$$

For the proof, let $\mathscr{D}(\Omega) \ni u_m \to \xi u$ in F. Then for $\phi \in \mathscr{D}(\Omega)$,

$$\begin{aligned}
\langle\langle \xi\eta\partial_t u, \phi\rangle\rangle &= \langle\langle \partial_t(\xi\eta u), \phi\rangle\rangle - \langle\langle (\partial_t\eta)\, \xi u, \phi\rangle\rangle \\
&= -\langle\langle \xi\eta u, \partial_t\phi\rangle\rangle - \langle\langle (\partial_t\eta)\, \xi u, \phi\rangle\rangle \\
&= -\iint_\Omega \xi u \partial_t(\eta\bar{\phi}) = -\lim_{m\to\infty} \iint_\Omega u_m \partial_t(\eta\bar{\phi}) \\
&= \lim_{m\to\infty} \langle\langle \partial_t u_m, \eta\phi\rangle\rangle_* = \langle\langle \xi\partial_t u, \eta\phi\rangle\rangle_* \\
&= \langle\langle \xi\partial_t u, \eta\phi\rangle\rangle
\end{aligned}$$

where the subscript $*$ indicates the $H^{-1}((0, T); \mathscr{H}^{-1}) - H_0^1((0, T); \mathscr{H}_0^1)$ pairing. This proves (6.3), which implies the estimate

$$|\langle\langle \sigma\partial_t u, \phi\rangle\rangle| \leqslant \text{Const}\, \|\phi\|_F,$$

and consequently $\sigma\partial_t u \in F^*$. ∎

Define

$$\mathscr{A} = \{u \in F \colon \partial_t u \in L^2((0, T); \mathscr{H})\}. \tag{6.4}$$

Obviously $\mathscr{A}$ is a subspace of $\mathscr{B}$. Give $\mathscr{A}$ the topology of $\mathscr{B}$.

6.2. LEMMA. *Assume Hypothesis 5. Then $\mathscr{A}$ is dense in $\mathscr{B}$.*

Proof. Since $\mathscr{B}$ is given by (6.1), the methods of [7, pp. 11–13] show that $C^\infty([0, T]; \mathscr{H}_0^1)$ is dense in $\mathscr{B}$. But $\mathscr{A} \supset C^\infty([0, T]; \mathscr{H}_0^1)$, so the proof is complete. ∎

We now turn to the proof of Theorem 5.3, Green's formula. The proof depends on the following result.

6.3. TRACE THEOREM. *Assume Hypotheses* 2–5.

(a) *The maps*

$$u \longmapsto |\sigma(\cdot, 0)|^{1/2}\, u(\cdot, 0), \qquad u \longmapsto |\sigma(\cdot, T)|^{1/2}\, u(\cdot, T)$$

are well-defined and continuous from $\mathscr{A}$ to $\mathscr{H}$.

(b) *There is a positive constant C such that*

$$\int_a^b |\sigma(x, 0)|\, |u(x, 0)|^2\, dx \leqslant C\, \|u\|_{\mathscr{B}}^2,$$

$$\int_a^b |\sigma(x, T)|\, |u(x, T)|^2\, dx \leqslant C\, \|u\|_{\mathscr{B}}^2$$

hold for all $u \in \mathscr{B}$.

This theorem will be proved in Sections 7 and 8. Assuming its validity we can finally complete the proof of Theorem 5.3.

Proof of (a). Let $u \in \mathscr{B}$. By Lemma 6.2, choose a sequence $\{u_n\}$ in $\mathscr{A}$ converging to u. By Part (a) of the trace theorem, $|\sigma(\cdot, t)|^{1/2}\, u_n(\cdot, t)$ for $t = 0, T$ exist and converge to well-defined elements of $\mathscr{H}$, which we denote by $|\sigma(\cdot, t)|^{1/2}\, u(\cdot, t)$ for $t = 0, T$. Since σ is bounded on $\bar{\Omega}$, so is $|\sigma|^{1/2}$, and the desired result follows.

Proof of (b). Since for $u, v \in \mathscr{A}$ the left hand side of equation (5.1) equals

$$\int_a^b \int_0^T \partial_t(\sigma u)\, \bar{v} + \int_a^b \int_0^T \sigma(\partial_t \bar{v})\, u = \int_a^b \left[\int_0^T \partial_t(\sigma u \bar{v})\, dt\right] dx,$$

we see that (5.1) is valid for $u, v \in \mathscr{A}$.

According to the trace theorem, the right hand side of (5.1) is a continuous function on $\mathscr{A} \times \mathscr{A}$, and by Lemma 6.2 it extends by continuity to $\mathscr{B} \times \mathscr{B}$. To complete the proof we must show that the left hand side of (5.1) also extends by continuity to $(u, v) \in \mathscr{B} \times \mathscr{B}$. But for $u, v \in \mathscr{B}$,

$$\begin{aligned}
|\langle\langle \partial_t(\sigma u), v\rangle\rangle| &\leqslant \|\sigma \partial_t u + (\partial_t \sigma)\, u\|_{F^*} \|v\|_F \\
&\leqslant (\|\sigma\partial_t u\|_{F^*} + |||(\partial_t\sigma)\, u|||)\, \|v\|_F \\
&\qquad \text{where } |||\cdot||| \text{ is the } L^2((0,T); \mathscr{H}) \text{ norm} \\
&\leqslant (\|\sigma\partial_t u\|_{F^*} + \text{Const}\, \|u\|_F)\, \|v\|_{\mathscr{B}} \\
&\leqslant \text{Const}\, \|u\|_{\mathscr{B}} \|v\|_{\mathscr{B}}
\end{aligned}$$

(see (3.10)). The other term $\overline{\langle\langle \sigma\partial_t v, u\rangle\rangle}$ satisfies a similar estimate by similar reasoning. This completes the proof. ∎

7. Proof of the Trace Theorem: Part I

The trace theorem is an immediate consequence of the following result.

7.1. Theorem. *Let Hypotheses 2–5 hold. There is a constant $C > 0$ such that for all $u \in \mathscr{A}$,*

$$\int_J |\sigma(x,t)|\, |u(x,t)|^2\, dx \leqslant C\, \|u\|_{\mathscr{B}}^2 \tag{7.1}$$

holds for $t = 0, T$ and $J = L, R$.

Proof. These integrals make sense because $u \in \mathscr{A}$. It is enough to prove the existence of a C such that

$$\int_0^b \sigma(x,0)\, |u(x,0)|^2\, dx \leqslant C\, \|u\|_{\mathscr{B}}^2 \tag{7.2}$$

for all $u \in \mathscr{A}$, since the same proof will apply with only minor changes to establish the other three estimates.

Let $\varepsilon_0 > 0$ be a small given number and let, for $i = 1, 2$, $\phi_i \in C^2(\mathbb{R}_+ \times \mathbb{R}_+; \mathbb{R})$ satisfy

$$\phi_1(x,t) = 0 \qquad \text{if} \quad t \geqslant T/2 \text{ or if } x \geqslant b/2 + \varepsilon_0, \tag{7.3}$$

$$\phi_2(x,t) = 0 \qquad \text{if} \quad x \leqslant b/4 + \varepsilon_0, \tag{7.4}$$

$$\phi_1(x,0) + \phi_2(x,0) = 1 \qquad \text{for} \quad x \in [0,b]. \tag{7.5}$$

For definiteness, choose b_1 as in Hypothesis 2 and take $0 < \varepsilon_0 < b_1/4$. By making b_1 smaller if necessary we may assume $b_1 \leqslant b/4$.

Let $u \in \mathscr{A}$. Then for all $t \in [0, T)$, $u(\cdot, t)$ is a well defined member of $\mathscr{H}$; thus $u(x, t)$ is defined for a.e. $x \in I$. Define

$$v_1 : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{C} \qquad \text{and} \qquad v_2 : \Omega_0 \equiv R \times (0, T) \to \mathbb{C}$$

by

$$v_1 = \phi_1 u \chi_{\Omega_0}, \qquad v_2 = \phi_2 u,$$

χ_{Ω_0} being the characteristic function of Ω_0. Then

$$v_1 \in L^2((0, T); H^1(R)), \qquad v_2 \in L^2((0, T); H_0^1(R)),$$
$$\partial_t v_1, \partial_t v_2 \in L^2((0, T); L^2(R)) \subset L^2((0, T); H^{-1}(R)).$$

By (7.5),

$$\int_0^b \sigma(x, 0)\, |u(x, 0)|^2 \, dx \leqslant 2 \sum_{i=1}^{2} \int_0^b \sigma(x, 0)\, |v_i(x, 0)|^2 \, dx. \tag{7.6}$$

Thus to prove (7.2) it suffices to prove it with u replaced by v_i on the left hand side for $i = 1, 2$. We split the proof into two parts, A and B.

Part A

We shall use the notation $H = H^0 \equiv L^2(R)$, $H_0^1 = H_0^1(R)$, $H^{\pm 1} = H^{\pm 1}(R)$, $\mathscr{H}^j = L^2((0, T); H^j)$ with or without a zero subscript. We shall let $\|\cdot\|_j$ and $|||\cdot|||_j$ denote the norms in H^j, $\mathscr{H}^j$ respectively. We must establish the existence of a constant C such that

$$\int_0^b \sigma(x, 0)\, |v_2(x, 0)|^2 \, dx \leqslant C \, \|u\|_{\mathscr{D}}^2 \tag{7.7}$$

holds for all $u \in \mathscr{A}$. Since σ is bounded,

$$\int_0^b \sigma(x, 0)\, |v_2(x, 0)|^2 \, dx \leqslant \text{Const}\, \|v_2(\cdot, 0)\|_0.$$

Next,

$$\|v_2(\cdot, 0)\|_0 \leqslant \text{Const}(|||v_2|||_1 + |||\partial_t v_2|||_{-1})$$

by a standard trace theorem (see [8, pp. 41–42]). So it suffices to show

$$|||v_2|||_1 \leqslant \text{Const}\, \|u\|_F, \tag{7.8}$$

$$|||\partial_t v_2|||_{-1} \leqslant \text{Const}\, \|u\|_{\mathscr{D}} \tag{7.9}$$

since the $L^2((0,T);H^{-1})$ norm is smaller than the $\mathscr{B}$ norm. The estimate (7.8) is easy:

$$|||v_1|||_1^2 = \int_0^T \int_0^b |\partial_x v_2(x,t)|^2\,dx\,dt$$

$$\leqslant \text{Const} \iint_{\Omega_0} \{|u|^2 + |\partial_x u|^2\} \leqslant \text{Const}\, \|u\|_F^2$$

because $\phi_2, \partial_x \phi_2 \in L^\infty(\Omega_0)$.

Let u_r be the restriction of u to Ω_0. We shall show that

$$\partial_t u_r \in \mathscr{H}^0, \tag{7.10}$$

$$|||\partial_t v_2|||_{-1}^2 \leqslant \text{Const}(\|u\|_F^2 + |||\phi_2 \partial_t u_r|||_{-1}^2) \tag{7.11}$$

and, for a.e. $t \in [0,T)$,

$$\|(\phi_2 \partial_t u_r)(t)\|_{-1} \leqslant \text{Const}\, |||(\sigma \partial_t u)(t)|||_{-1}. \tag{7.12}$$

Since (7.11) and (7.12) imply (7.9), proving (7.10)–(7.12) will complete the proof of (7.7).

For (7.10) first take $\omega \in \mathscr{D}(\Omega_0)$ and extend ω to be zero elsewhere in Ω. Then $\langle\langle \partial_t u_r, \omega \rangle\rangle = \iint_{\Omega_0} (\partial_t u_r)\,\bar{\omega} = \iint_\Omega (\partial_t u)\,\bar{\omega}$, whence

$$|\langle\langle \partial_t u_r, \omega \rangle\rangle| \leqslant \|\partial_t u\|_{L^2((0,T);\mathscr{H})}\, |||\omega|||_0,$$

so, by the density of $\mathscr{D}(R \times (0,T))$ in $\mathscr{H}_0$, (7.10) follows. Next,

$$\iint_{\Omega_0} (\partial_t u_r)\,\bar{\omega} = \iint_{\Omega_0} (\partial_t u)\,\bar{\omega}$$

for all $\omega \in \mathscr{D}(\Omega_0)$, so

$$\partial_t u_r(x,t) = \partial_t u(x,t) \qquad \text{for} \quad \text{a.e. } (x,t) \in \Omega_0. \tag{7.13}$$

Inequality (7.11) is an easy consequence of (7.13) and (7.9) because $\partial_t v_2 = (\partial_t \phi_2)\, u_r + \phi_2 (\partial_t u_r)$ in Ω_0. For Part A it remains to prove (7.12).

Using (7.13) we see that for a.e. $t \in [0,T]$,

$$\|(\phi_2 \partial_t u_r)(t)\|_{-1}$$
$$= \sup \left\{ \left| \int_{b/4+\varepsilon_0}^b \phi_2(x,t)\, \partial_t u_r(x,t)\, \bar{y}(x)\, dx \right| : y \in H_0^1(R), \|y\|_1 = 1 \right\}.$$

For $x \in [b/4 + \varepsilon_0, b]$, by Hypothesis 2 we can write

$$\phi_2(x,t) = \sigma(x,t)(\phi_2(x,t)/\sigma(x,t))$$

and we can extend ϕ_2/σ to be zero elsewhere on Ω; call this extension θ. Then $\theta, \partial_x\theta \in L^\infty(\Omega)$. Thus for

$$Y = \{y \in H_0^1(I): \|y\|_1 = 1, y(x) = 0 \text{ for all } x \in L\},$$

$$\begin{aligned}\|(\phi_2\partial_t u_r)(t)\|_{-1} &= \sup\left\{\left|\int_a^b \sigma(\partial_t u)\,\theta\bar{y}\,dx\right| : y \in Y\right\}\\ &\leqslant \|(\sigma\partial_t u)(t)\|_{-1} \sup\{\|\theta(\cdot, t)\,y\|_1 : y \in Y\}\\ &\leqslant \text{Const}\,\|(\sigma\partial_t u)(t)\|_{-1},\end{aligned}$$

which is (7.12). This completes Part A.

Part B

Now we prove

$$\int_0^b \sigma(x, 0)\,|v_1(x, 0)|^2\,dx \leqslant \text{Const}\,\|u\|_{\mathscr{F}}^2; \tag{7.14}$$

then the proof of (7.2) will be complete. The method of Part A will not work here since v_1 need not vanish in a neighborhood of $x = 0$.

Let K be the Hilbert space

$$K = \{v \in L^2(\mathbb{R}_+; H^1(\mathbb{R}_+)): \sigma_+\partial_t v \in L^2(\mathbb{R}_+; H^{-1}(\mathbb{R}_+))\}$$

whose norm is given by

$$\|v\|_K = \{\|v\|^2_{L^2(\mathbb{R}_+;L^2(\mathbb{R}_+))} + \|\partial_x v\|^2_{L^2(\mathbb{R}_+;L^2(\mathbb{R}_+))} + \|\sigma_+\partial_t v\|^2_{L^2(\mathbb{R}_+;H^{-1}(\mathbb{R}_+))}\}^{1/2}.$$

We first want to show that $v_1 \in K$ and to get an estimate on $\|v_1\|_K$. Since v_1 vanishes outside $\Omega_1 = [0, b/2 + \varepsilon_0] \times [0, T)$, it is clear that

$$\int_0^\infty \int_0^\infty (|v_1|^2 + |\partial_x v_1|^2) \leqslant \text{Const}\,\|u\|_F^2 < \infty. \tag{7.15}$$

Thus $v_1 \in L^2(\mathbb{R}_+; H^1(\mathbb{R}_+))$ and

$$\partial_t v_1(x, t) = ((\partial_t\phi_1)(x, t)\,u(x, t) + \phi_1(x, t)\,\partial_t u(x, t))\,\chi_{\Omega_1}(x, t) \text{ a.e.}.$$

The various assumptions on σ, ϕ_1, and u enable us to deduce that

$$\|(\sigma_+\partial_t v_1)(t)\|_{H^{-1}(\mathbb{R}_+)} = 0 \qquad \text{for} \quad t \geqslant T/2 \tag{7.16}$$

and for a.e. $t \in [0, T/2]$,

$$\int_0^\infty |\partial_t v_1(x,t)|^2 \, dx = \int_0^{(b/2)+\varepsilon_0} |\partial_t v_1|^2 \, dx < \infty,$$

$$\int_0^\infty |\sigma_+ \partial_t v_1|^2 \, dx = \int_0^{(b/2)+\varepsilon_0} |\sigma \partial_t v_1|^2 \, dx < \infty,$$

$$(\sigma_+ \partial_t v_1)(t) \in L^2(\mathbb{R}_+) \subset H^{-1}(\mathbb{R}_+).$$

Note that if $y \in H_0^1(\mathbb{R}_+)$ and y vanishes on R, then

$$\langle \sigma_+ \partial_t v_1, y \rangle = 0$$

where this is the $H^{-1}(\mathbb{R}_+) - H_0^1(\mathbb{R}_+)$ pairing. Consequently for a.e. $t \in [0, T/2]$,

$$\|(\sigma_+ \partial_t v_1)(t)\|_{-1} = \sup \left\{ \left| \int_0^b (\sigma \partial_t v)(x,t) \, \overline{y(x)} \, dx \right| : y \in Y_1 \right\} \tag{7.17}$$

where here and below $\|\cdot\|_j$ refers to the $H^j(\mathbb{R}_+)$ norm (or the $H^j(I)$ norm) and

$$Y_1 = \{ y \in H_0^1(\mathbb{R}_+) : \|y\|_1 = 1 \}.$$

Next choose $\theta \in C(\overline{\mathbb{R}}_+; \overline{\mathbb{R}}_+)$ so that $\theta(x) = 1$ for $x \leqslant b/2 + \varepsilon_0$ and $\theta(x) = 0$ for $x \geqslant b/2 + 2\varepsilon_0$. If $y \in Y_1$, then $\theta y \in \mathscr{H}_0^1$ and

$$\begin{aligned} \left| \int_0^b (\sigma \partial_t v_1) \, \bar{y} \, dx \right| &\leqslant \left| \int_0^b u(\sigma \partial_t \phi_1) \, \theta \bar{y} \, dx \right| + \left| \int_0^b (\sigma \partial_t u) \, \phi \theta \bar{y} \, dx \right| \\ &\leqslant \text{Const} \, \|u(\cdot, t)\|_0 \, \|y\|_0 + \|\sigma \partial_t u\|_{-1} \, \|\phi_1(\cdot, t) \, \theta y\|_1. \end{aligned} \tag{7.18}$$

Since $\|y\|_0 \leqslant \|y\|_1$ and $\|\phi_1(\cdot, t) \, \theta y\|_1 \leqslant \text{Const} \, \|y\|_1$, we can plug (7.18) into (7.17) to obtain

$$\|\sigma_+ \partial_t v(t)\|_{-1} \leqslant \text{Const}(\|u(\cdot, t)\|_1 + \|(\sigma \partial_t u)(t)\|_{-1})$$

for a.e. $t \in [0, T/2]$. This, together with (7.15) and (7.16) yields

$$\|v_1\|_K^2 \leqslant \text{Const} \, \|u\|_{\mathscr{B}}^2.$$

In the next section we shall establish that for all $v_1 \in K$,

$$\int_0^\infty |\sigma_+(x, 0)| \, |v_1(x, 0)|^2 \, dx \leqslant \text{Const} \, \|v_1\|_K^2. \tag{7.19}$$

Assuming the validity of this estimate we conclude that, for $v_1 = \phi_1 u \chi_{\Omega_0}$,

$$\int_0^b |\sigma(x,0)|\,|v_1(x,0)|^2\,dx \leqslant \int_0^\infty |\sigma(x,0)|\,|v_1(x,0)|^2\,dx$$

$$\leqslant \text{Const}\,\|v_1\|_K^2 \leqslant \text{Const}\,\|u\|_{\mathscr{B}}^2$$

(see (7.2)), and the theorem is proved.

8. Proof of the Trace Theorem: Part II

We shall employ the following notational convention. If $u\colon \overline{\mathbb{R}}_+ \to X$ is a function, then its even extension to all of $\mathbb{R}$ will be denoted by $\tilde{u}$. Thus (see Hypothesis 3) $\sigma_0 = \tilde{\sigma}_1$. The term mollifier will refer to a sequence $\{\rho_m\}$ in $\mathscr{D}(\mathbb{R})$ of nonnegative even functions satisfying $\int_{-\infty}^{\infty} \rho_m(t)\,dt = 1$ and $\operatorname{supp} \rho_m \subset [-\beta_m, \beta_m]$ where $\beta_m \to 0$. In what follows ρ_m will always be a function of the time variable t.

Define the Hilbert space W to be

$$W = \{u \in L^2(\mathbb{R}_+; H^1(\mathbb{R}));\, |\sigma_1|\,\partial_t u \in L^2(\mathbb{R}_+; H^{-1}(\mathbb{R}))\},$$

the norm of which is given by

$$\|u\|_W = \{|||u|||_1^2 + |||\,|\sigma_1|\,\partial_t u|||_{-1}^2\}^{1/2}$$

where, in this section, $|||\cdot|||_j$ denotes the norm in either $\mathscr{H}_j = L^2(\mathbb{R}; H^j(\mathbb{R}))$ or $\mathscr{H}_j^+ = L^2(\mathbb{R}_+; H^j(\mathbb{R}))$.

From now on Hypotheses 2–5 will be assumed.

8.1. Lemma. *Let $u \in W$. Then*

(i) $\tilde{u} \in \mathscr{H}_1$,

(ii) $|\sigma_0|\,\partial_t \tilde{u} \in \mathscr{H}_{-1}$,

(iii) $(|\sigma_1|\,\partial_t u)(t) = |\sigma_0|\,\partial_t \tilde{u}(t)$ *for a.e.* $t > 0$.

Proof. (i) is obvious, so we proceed to (ii). For a.e. $\tau \in \mathbb{R}$, $(|\sigma_0|\,\partial_t \tilde{u})(\tau) \in H^{-1}(\mathbb{R})$. For all $\phi \in H^1(\mathbb{R})$,

$$\langle (|\sigma_0|\,\partial_t \tilde{u})(\tau), \phi\rangle = \langle \partial_t(|\sigma_0|\,\tilde{u})(\phi), \phi\rangle - \langle ((\partial_t |\sigma_0|)\,\tilde{u})(\tau), \phi\rangle.$$

Using $(\partial_t w)(\tau) = \lim_{\Delta\tau \to 0} (\Delta\tau)^{-1}(w(\tau + \Delta\tau) - w(\tau))$, we conclude

$$\begin{aligned} \langle (|\sigma_0|\,\partial_t \tilde{u})(\tau), \phi\rangle &= \langle (|\sigma_1|\,\partial_t u)(\tau), \phi\rangle \qquad &\text{if} \quad \tau > 0, \\ &= -\langle (|\sigma_1|\,\partial_t u)(-\tau), \phi\rangle \qquad &\text{if} \quad \tau < 0, \end{aligned}$$

whence (ii) follows. From this (iii) also follows. ∎

Let S_m denote the convolution operator

$$S_m v = \rho_m * v$$

where the convolution is in the time variable t and the mollifier $\{\rho_m\}$ is given.

8.2. Lemma. *$W \cap C^\infty(\bar{\mathbb{R}}_+; H^1(\mathbb{R}))$ is dense in W.*

Proof. Let $u \in W$. Then $\tilde{u} \in \mathscr{H}_1$. Then (cf. [1, 8]),

$$S_m \tilde{u} \to \tilde{u} \qquad \text{in } \mathscr{H}_1 \tag{8.1}$$

as $m \to \infty$. Since $\sigma_0 \partial_t \tilde{u} \in \mathscr{H}_{-1}$ we can show as in Lemma 6.1 that $\xi \partial_t \tilde{u} \in \mathscr{H}_{-1}$ (see Hypothesis 5). This implies

$$\xi S_m \partial_t \tilde{u} = S_m(\xi \partial_t \tilde{u}) \to \xi \partial_t \tilde{u} \tag{8.2}$$

in $L^2(\mathbb{R}; H^{-1}(\mathbb{R}))$ as $m \to \infty$. Multiplying (8.2) by η leads to

$$\sigma_0 S_m \partial_t \tilde{u} \to \sigma_0 \partial_t \tilde{u} \qquad \text{in } \mathscr{H}_{-1} \tag{8.3}$$

as $m \to \infty$, which we shall prove presently. From (8.1), (8.3), and Lemma 8.1(iii), it follows that the restriction of $S_m \tilde{u}$ to $\mathbb{R}_+$ converges to u in W. Since $S_m \tilde{u} \in C^\infty(\mathbb{R}; H^1(\mathbb{R}))$, the restriction of $S_m \tilde{u}$ to $\bar{\mathbb{R}}_+$ is in $C^\infty(\bar{\mathbb{R}}_+; H^1(\mathbb{R}))$. Thus it only remains to prove (8.3).

Note that $\xi S_m(\partial_t \tilde{u}) = S_m(\partial_t(\xi \tilde{u}))$ is in $C^\infty(\mathbb{R}; H^1(\mathbb{R}))$. Consequently $\sigma_0 S_m(\partial_t \tilde{u}) \in C^1(\mathbb{R}; H^1(\mathbb{R})) \subset \mathscr{H}_1$. Using the methods of the proof of Lemma 6.1 we argue as follows. For all $\phi \in \mathscr{D}(\mathbb{R}^2)$,

$$\langle\!\langle \sigma_0 S_m(\partial_t \tilde{u}) - \sigma_0 \partial_t \tilde{u}, \eta^{-1}\phi \rangle\!\rangle = \langle\!\langle \xi(S_m(\partial_t \tilde{u}) - \partial_t \tilde{u}), \phi \rangle\!\rangle. \tag{8.4}$$

Given $v \in \mathscr{H}_1$ let $\{\phi_n\}$ be a sequence in $\mathscr{D}(\mathbb{R}^2)$, supported in $J \times \mathbb{R}$ for some bounded spatial interval J, and which converges to v in $\mathscr{H}_1$. Thus (8.4) yields

$$\langle\!\langle \sigma_0(S_m(\partial_t \tilde{u}) - \partial_t \tilde{u}), \eta^{-1} v \rangle\!\rangle = \langle\!\langle \xi(S_m(\partial_t \tilde{u}) - \partial_t \tilde{u}), v \rangle\!\rangle$$

for $v \in \mathscr{H}_1$. So for all $v \in \mathscr{H}_1$,

$$\langle\!\langle \sigma_0(S_m(\partial_t \tilde{u}) - \partial_t \tilde{u}), v \rangle\!\rangle = \langle\!\langle \xi(S_m(\partial_t \tilde{u}) - \partial_t \tilde{u}), \eta v \rangle\!\rangle,$$

whence

$$\begin{aligned} |\!|\!| \sigma_0 S_m(\partial_t \tilde{u}) - \sigma_0 \partial_t \tilde{u} |\!|\!|_{-1} &\leqslant \text{Const}\, |\!|\!| \xi(S_m(\partial_t \tilde{u}) - \partial_t \tilde{u}) |\!|\!|_{-1} \\ &\to 0 \qquad \text{as } \quad m \to \infty \text{ by (8.2).} \quad \blacksquare \end{aligned}$$

Let $\mathscr{E}'$ consist of those distributions in $\mathscr{D}'(\mathbb{R} \times \mathbb{R}_+)$ which have bounded support.

8.3. LEMMA. *$V \equiv W \cap C^\infty(\bar{\mathbb{R}}_+; H^1(\mathbb{R})) \cap \mathscr{E}'$ is dense in W.*

Proof. Let $u \in W$ and let $u_m \in C^\infty(\mathbb{R} \times \bar{\mathbb{R}}_+)$ be the restriction of $S_m \tilde{u}$ to $\bar{\mathbb{R}}_+$. By Lemma 8.2 it is enough to show that each u_m is in the closure of V in the W topology.

Choose $\phi \in C_0^\infty(\mathbb{R}; \mathbb{R})$ such that ϕ is even, $0 \leqslant \phi \leqslant 1$, and $\phi(z) = 1$ (resp. $=0$) for $|z| \leqslant 1$ (resp. for $|z| \geqslant 2$). Let M_0 be an upper bound for $|\phi'(z)|$ on R. Define $u_{mn}: \mathbb{R} \times \mathbb{R}_+ \to C$ by

$$u_{m,n}(x, t) = \phi(t/n)\, \phi(x/n)\, u_m(x, t). \tag{8.5}$$

It remains to show that $u_{mn} \in V$ and $u_{mn} \to u_m$ in W as $n \to \infty$.

For notational ease we shall write ϕ_n (resp. ψ_n) as the function whose value at $x \in \mathbb{R}$ (resp. $t \in \mathbb{R}_+$) in $\phi(x/n)$ (resp. $\phi(t/n)$). Thus (8.5) says $u_{mn} = \psi_n \phi_n u_m$. Since $\|\phi\|_\infty \leqslant 1$ and $u_m \in W \subset \mathscr{H}_1^+$, (8.5) implies $u_{mn} \in \mathscr{H}_0^+$. We clearly have

$$\partial_x u_{mn} = n^{-1} \psi_n(\partial_x \phi_n)\, u_m + \psi_n \phi_n \partial_x u_m, \tag{8.6}$$

$$\partial_t u_{mn} = n^{-1} (\partial_t \psi_n)\, \phi_n u_m + \psi_n \phi_n \partial_t u_m. \tag{8.7}$$

The right hand side of (8.6) is in $\mathscr{H}_0^+$, so it follows that $u_{mn} \in \mathscr{H}_1^+$. To show $u_{mn} \in V$, we first show that $|\sigma_1|\, \partial_t u_{mn} \in \mathscr{H}_{-1}^+$. For this, look at the first term on the right hand side of (8.7).

$$\begin{aligned} |||n^{-1}\, |\sigma_1| (\partial_t \psi_n)\, \phi_n u_m|||_{-1}^2 &\leqslant |||n^{-1}\, |\sigma_1| (\partial_t \psi_n)\, \phi_n u_m|||_0^2 \\ &\leqslant M_0^2 n^{-2} \int_0^{2n} \int_{-2n}^{2n} |\sigma_1(x, t)|^2\, |u_m(x, t)|^2\, dx\, dt \\ &\leqslant \text{Const}\, |||u_m|||_0^2 < \infty \end{aligned}$$

by Hypothesis 3. Thus $n^{-1}\, |\sigma_1| (\partial_t \psi_n)\, \phi_n u_m \in \mathscr{H}_0^+$. Next, $u_m \in C^\infty(\bar{\mathbb{R}}_+; H^1(\mathbb{R}))$, so that Hypothesis 3 implies $|\sigma_1|\, \psi_n \phi_n \partial_t u_m \in \mathscr{H}_1^+$. Thus from (8.7) we deduce $|\sigma_1|\, \partial_t u_{mn} \in \mathscr{H}_0^+ \subset \mathscr{H}_{-1}^+$. This proves $u_{mn} \in W$.

Since ϕ and u_m are C^∞ functions, (8.5) implies $u_{mn} \in C^\infty(\bar{\mathbb{R}}_+; H^1(\mathbb{R}))$. Next let $\chi \in \mathscr{D}(\mathbb{R} \times \mathbb{R}_+)$ be supported outside $[-2n, 2n] \times (0, 2n]$. Then it is easy to check that $\langle\langle u_{mn}, \chi \rangle\rangle = 0$, whence $u_{mn} \in \mathscr{E}'$. Thus $u_{mn} \in V$. It only remains to show that $u_{mn} \to u_m$ in W as $n \to \infty$.

Since $\phi(z/n) \to 1$ as $n \to \infty$ for each $z \in \mathbb{R}$, (8.5) and (8.6) imply $u_{mn} \to u_n$ pointwise and $\partial_x u_{mn} \to \partial_x u_m$ a.e. as $n \to \infty$. Lebesgue's dominated convergence theorem, (8.5) and $\|\phi\|_\infty \leqslant 1$ imply $u_{mn} \to u_m$ in $\mathscr{H}_1^+$ as $n \to \infty$. It now only remains to show that $|\sigma_1|\, \partial_t u_{mn} \to |\sigma_1|\, \partial_t u_m$ in $\mathscr{H}_{-1}^+$ as $n \to \infty$.

Equation (8.7) and Hypothesis 3 imply $|\sigma_1|\, \partial_t u_{mn} \to |\sigma_1|\, \partial_t u_m$ a.e. Moreover, $|(|\sigma_1|\, \partial_t u_{mn})(x, t)| \leqslant M_0 n^{-1}\, |(\sigma_1 u_m)(x, t)| + |(|\sigma_1|\, \partial_t u_m)(x, t)|$, and

$|\sigma_1|\,\partial_t u_m$, $|\sigma_1|\,u_m \in \mathscr{H}_0^+$. Lebesgue's dominated convergence theorem yields $|\sigma_1|\,\partial_t u_{mn} \to |\sigma_1|\,\partial_t u_m$ in $\mathscr{H}_0^+$ and therefore in $\mathscr{H}_{-1}^+$ as $n \to \infty$. ∎

8.4. LEMMA. *The linear operator $L_0\colon u \mapsto |\sigma_1(\cdot, 0)|^{1/2}\, u(\cdot, 0)$ is well-defined and continuous from V to $L^2(\mathbb{R})$ where V is given the W topology.*

Proof. The lemma implies the existence of a constant such that for all $u \in V$ (hence all $u \in W$),

$$\int_{-\infty}^{\infty} |\sigma_1(x, 0)|\, |u(x, 0)|^2\, dx \leqslant \text{const}\, \|u\|_W^2. \tag{8.8}$$

For the proof, let $u \in V$. Then $u(x, t)$, $\partial_t u(x, t)$ are defined for all $(x, t) \in \mathbb{R} \times \overline{\mathbb{R}}_+$. Since u has bounded support ($u \in \mathscr{E}'$), u vanishes outside a bounded rectangle $[v_1, v_2] \times [0, T_1]$; thus $\int_{-\infty}^{\infty} |\sigma_1(x, t)|\, |u(x, t)|^2\, dx$ is well-defined and finite for each $t \geqslant 0$. In particular, L_0 makes sense.

$$\begin{aligned}
\int_{-\infty}^{\infty} |L_0 u(x)|^2\, dx &= \int_{v_1}^{v_2} |\sigma_1(x, 0)|\, |u(x, 0)|^2\, dx \\
&= -\int_0^{2T_1} \frac{d}{dt} \int_{v_1}^{v_2} |\sigma_1(x, t)|\, |u(x, t)|^2\, dx\, dt \\
&= -\int_0^{2T_1} \int_{v_1}^{v_2} \{(\partial_t |\sigma_1|)\, |u|^2 + |\sigma_1|\, 2\, \mathrm{Re}[(\partial_t u)\, \bar{u}]\}\, dx\, dt;
\end{aligned}$$

the differentiation under the integral sign is legitimate by Hypothesis 3. Thus

$$\begin{aligned}
\int_{-\infty}^{\infty} |\sigma_1(x, 0)|\, |u(x, 0)|^2\, dx &\leqslant \text{Const}[|||u|||_0^2 + |||\, |\sigma_1|\, \partial_t u\, |||_{-1}\, |||u|||_1] \\
&\leqslant \text{Const}\, \|u\|_W^2
\end{aligned}$$

by Hypothesis 3 again, and the lemma follows. ∎

Similarly the map $L_1 : u \mapsto u(\cdot, 0)$ can be shown to be continuous from V to $L^2(\mathbb{R})$.

The next result generalizes the Baouendi–Grisvard extension theorem [2, p. 363]. Recall the space K defined in the paragraph containing (7.15).

8.5. EXTENSION THEOREM. *There is a continuous linear operator $\mathscr{P}\colon K \to W$ and a constant C_0 such that for all $v \in K$, $\mathscr{P}v = v$ when the spatial variable is restricted to lie in $\mathbb{R}_+$ and*

$$\int_{-\infty}^{\infty} |\sigma_1(x, 0)|^2\, |\mathscr{P}v(x, 0)|^2\, dx \leqslant C_0\, \|\mathscr{P}v\|_W^2 \leqslant C_0^2\, \|v\|_K^2.$$

Proof. By Lemma 8.4 it is enough to establish the existence of a bounded linear extension operator $\mathscr{P}$ from K to W. To that end, define

$$\mathscr{P}: L^2(\mathbb{R}_+; H^1(\mathbb{R}_+)) \to L^2(\mathbb{R}_+; H^1(\mathbb{R})) \tag{8.9}$$

by

$$\begin{aligned}(\mathscr{P}u)(x, t) &= u(x, t) && \text{for} \quad x \geqslant 0, t > 0,\\ &= \sum_{j=1}^{2} \alpha_j(x)\, u(g_j(x), t) && \text{for} \quad x < 0, t > 0,\end{aligned}$$

where the notation introduced in Hypothesis 4 is being used. $\mathscr{P}$ is well-defined; indeed, if $u \in \mathscr{K}_1^+ = L^2(\mathbb{R}_+; H^1(\mathbb{R}_+))$, then by Hypothesis 4,

$$\int_0^\infty \int_0^\infty |\mathscr{P}u(x, t)|^2\, dx\, dt = |||u|||_0^2 < \infty,$$

$$\int_0^\infty \int_0^\infty |\partial_x \mathscr{P}u(x, t)|^2\, dx\, dt \leqslant |||u|||_1^2 < \infty,$$

$$\begin{aligned}\int_0^\infty \int_{-\infty}^0 |\mathscr{P}u|^2\, dx\, dt &\leqslant \text{Const} \int_0^\infty \int_{-\infty}^0 \sum_{j=1}^{2} |u(g_j(x), t)|^2\, dx\\ &\leqslant \text{Const} \int_0^\infty \int_0^\infty \sum_{j=1}^{2} \mu^{-1}\, |u(y, t)|^2\, dy\, dt\\ &\leqslant \text{Const}\, |||u|||_0^2 < \infty,\end{aligned}$$

$$\begin{aligned}\int_0^\infty \int_{-\infty}^0 |\partial_x \mathscr{P}u|^2\, dx\, dt &\leqslant 16 \int_0^\infty \sum_{j=1}^{2} \int_{-\infty}^0 -\gamma_j(x)\, \partial_x g_j(x)\, |u(g_j(x), t)|^2\, dx\, dt\\ &\quad + 16 \int_0^\infty \sum_{j=1}^{2} \int_{-\infty}^0 |\alpha_j(x)|^2\, |\partial_x u(g_j(x), t)|^2\, g_j'(x)^2\, dx\, dt\\ &= 16 \sum_{j=1}^{2} \int_0^\infty \int_0^\infty [(u(y, t)^2\, \gamma_j(g_j^{-1}(y))\\ &\quad + |\alpha_j(g_j^{-1}(y))|^2\, |\partial_x g_j(g_j^{-1}(y))|\, |\partial_x u(y, t)|^2]\, dy\, dt\\ &\leqslant \text{Const}\, |||u|||_1^2 < \infty.\end{aligned} \tag{8.10}$$

Next we show that $\partial_x \mathscr{P}u$ defines a continuous functional on $\mathscr{K}_0^+$. For $\omega \in \mathscr{D}(\mathbb{R} \times \mathbb{R}_+)$,

$$\begin{aligned}
\langle\!\langle \partial_x \mathscr{P}u, \omega \rangle\!\rangle &= -\langle\!\langle \mathscr{P}u, \partial_x \omega \rangle\!\rangle \\
&= -\int_0^\infty \int_0^\infty u \partial_x \omega \, dx \, dt \\
&\quad - \int_0^\infty \int_{-\infty}^0 \sum_{j=1}^2 \alpha_j(x) \, u(g_j(x), t) \, \partial_x \omega(x, t) \, dx \, dt \\
&= \int_0^\infty u(0, t) \, \omega(0, t) \, dt + \int_0^\infty \int_0^\infty (\partial_x u) \, \omega \, dx \, dt \\
&\quad - \int_0^\infty \sum_{j=1}^2 \alpha_j(0) \, u(0, t) \, \omega(0, t) \, dt \\
&\quad + \int_0^\infty \int_{-\infty}^0 \sum_{j=1}^2 \partial_x [\alpha_j u(g_j, t)] \, \omega \, dx \, dt;
\end{aligned}$$

and since $\alpha_1(0) + \alpha_2(0) = 1$ we conclude that

$$|\langle\!\langle \partial_x \mathscr{P}u, \omega \rangle\!\rangle| \leqslant \text{Const} \, |||\omega|||_0 \, |||u|||_1$$

by a calculation similar to the one that led to (8.10). Thus $\mathscr{P}$ is well-defined and continuous (see (8.9)).

Next we show that $|\sigma_0| \, \partial_t(\mathscr{P}u) \in \mathscr{K}_{-1}$ for all $u \in K$ and that $\mathscr{P}: K \to W$ is continuous.

We start by defining a linear operator $\mathscr{Q}_1: \mathscr{K}_1 \to L^2(\mathbb{R}_+; H_0^1(\mathbb{R}_+))$ by

$$(\mathscr{Q}_1 u)(x, t) = u(x, t) + \sum_{j=1}^2 \beta_j(x, t) \, u(g_j^{-1}(x), t)$$

for $u \in \mathscr{K}_1$ (cf. Hypothesis 4). For $u \in \mathscr{K}_1$,

$$\begin{aligned}
\int_0^\infty \int_0^\infty |\mathscr{Q}_1 u|^2 \, dx \, dt &\leqslant 3 \int_0^\infty \int_0^\infty \left[|u|^2 + \sum_{j=1}^2 |\beta_j|^2 \, |u(g_j^{-1}(x), t)|^2 \right] dx \, dt \\
&\leqslant 3 \int_0^\infty \int_0^\infty |u|^2 \, dx \, dt \\
&\quad + 3 \int_0^\infty \int_{-\infty}^0 |\beta_j(g_j(y), t)|^2 \, |u(y, t)|^2 \, |g_j'(y)| \, dy \, dt \\
&\leqslant \text{Const} \, |||u|||_0^2
\end{aligned}$$

by Hypothesis 4. Thus the linear map $\mathscr{Q}_1$ is continuous as a map from $\mathscr{K}_1$ to $\mathscr{K}_0^+$. Next, for all $\phi \in \mathscr{D}(\mathbb{R}_+ \times \mathbb{R}_+)$,

$$\begin{aligned}\langle\langle \partial_x \mathcal{Q}_1 u, \phi\rangle\rangle &= \int_0^\infty \int_0^\infty (\partial_x u)\,\bar{\phi}\,dx\,dt \\ &\quad - \int_0^\infty \int_0^\infty \sum_{j=1}^{2} \{\partial_x \beta_j(x,t)\,u(g_j^{-1}(x),t) \\ &\quad + \beta_j(x,t)\,\partial_x u(g_j^{-1}(x),t)\,\partial_x g_j^{-1}(x)\}\,\bar{\phi}(x,t)\,dx\,dt \\ &\leqslant \text{Const}\,|||u|||_1\,|||\phi|||_0\end{aligned}$$

by Hypothesis 4. Thus $\mathcal{Q}_1$ is a continuous map from $\mathcal{K}_1$ to $\mathcal{K}_1^+$. By Hypothesis 4 we have $\sum_{j=1}^2 \alpha_j(0)/g_j'(0) = -1$ and so $1 + \sum_{j=1}^2 \beta_j(0,t) = 0$. This implies $(\mathcal{Q}_1 u)(0,t) = 0$ for all $t \geqslant 0$. Thus $(\mathcal{Q}_1 u)(\cdot,t) \in H_0^1(\mathbb{R}_+)$, and so $\mathcal{Q}_1$ is continuous as a map from $\mathcal{K}_1$ to $L^2(\mathbb{R}_+; H_0^1(\mathbb{R}_+))$.

Let now $v \in \mathcal{K}_1^+$. By Hypothesis 3, $(\partial_t|\sigma_1|)\,\mathcal{P}v$, $|\sigma_1|\,\mathcal{P}v \in \mathcal{K}_1$. Thus

$$|\sigma_1|\,\partial_t(\mathcal{P}v) = \partial_t(|\sigma_1|\,\mathcal{P}v) - (\partial_t(|\sigma_1|))\,\mathcal{P}v \in \mathcal{D}'(\mathbb{R} \times \mathbb{R}_+)$$

and for each $\phi \in \mathcal{D}(\mathbb{R} \times \mathbb{R}_+)$,

$$\langle\langle |\sigma_1|\,\partial_t(\mathcal{P}v), \phi\rangle\rangle = -\langle\langle (|\sigma_1|\,\mathcal{P}v, \partial_t\phi\rangle\rangle - \langle\langle (\partial_t|\sigma_1|)\,\mathcal{P}v, \phi\rangle\rangle. \tag{8.11}$$

We calculate separately the terms on the right hand side of (8.11). Using the definition of $\mathcal{P}$,

$$\begin{aligned}\langle\langle |\sigma_1|\,\mathcal{P}v, \partial_t\phi\rangle\rangle &= \int_0^\infty \int_0^\infty |\sigma_1|\,v\partial_t\bar{\phi}\,dx\,dt \\ &\quad + \int_0^\infty \int_{-\infty}^0 \sum_{j=1}^2 |\sigma_1(x,t)|\,\alpha_j(x)\,v(g_j(x),t)\,\partial_t\bar{\phi}(x,t)\,dx\,dt \\ &= \int_0^\infty \int_0^\infty (\sigma_1 v)(x,t)\left\{\partial_t\bar{\phi}(x,t)\right. \\ &\quad \left. + \sum_{j=1}^2 \beta_j(x,t)\,\partial_t\bar{\phi}(g_j^{-1}(x),t)\right\}dx\,dt\end{aligned} \tag{8.12}$$

by Hypotheses 3 and 4. Next,

$$\begin{aligned}&\langle\langle (\partial_t|\sigma_1|)\,\mathcal{P}v, \phi\rangle\rangle \\ &= \int_0^\infty \int_0^\infty v(\partial_t|\sigma_1|)\,\bar{\phi}\,dx\,dt \\ &\quad + \int_0^\infty \int_{-\infty}^0 \sum_{j=1}^2 \alpha_j(x)\,v(g_j(x),t)(\partial_t|\sigma_1|)\,\bar{\phi}\,dx\,dt\end{aligned}$$

$$= \int_0^\infty \int_0^\infty \Big\{ [v(\partial_t \sigma_1)\bar{\phi}](y,t)$$

$$+ \sum_{j=1}^{2} \alpha_j(g_j^{-1}(y))[g_j'(g_j^{-1}(y))]^{-1} v(y,t)[(\partial_t\sigma_1)\bar{\phi}](g_j^{-1}(y),t)\Big\}\, dy\, dt. \tag{8.13}$$

Substitute for β_j in (8.12) and then add the resulting equation to (8.13) keeping (8.11) in mind; the result is that for all $v \in \mathscr{K}_1^+$, $\phi \in \mathscr{D}(\mathbb{R} \times \mathbb{R}_+)$,

$$\langle\langle |\sigma_1|\, \partial_t(\mathscr{P}v), \phi\rangle\rangle = -\int_0^\infty \int_0^\infty [v\partial_t(\sigma_+(\mathscr{Q}_1\phi))](x,t)\, dx\, dt. \tag{8.14}$$

A short calculation, using the various boundedness hypotheses, shows that $\partial_t[\sigma_1 \mathscr{Q}_1 \phi] \in \mathscr{K}_0^+$. Consequently (8.14) yields that for all $v \in K$, all $\phi \in \mathscr{D}(\mathbb{R} \times \mathbb{R}_+)$, and any sequence $\{\omega_n\}$ in $C^\infty(\bar{\mathbb{R}}_+ \times \bar{\mathbb{R}}_+)$ converging to v in $\mathscr{K}_0^+$,

$$\begin{aligned}
\langle\langle |\sigma_1|\, \partial_t(\mathscr{P}v), \phi\rangle\rangle &= -\langle\langle v, \partial_t(\sigma_+ \mathscr{Q}_1\phi\rangle\rangle = -\lim_{n\to\infty} \int_0^\infty \int_0^\infty \omega_n \partial_t(\sigma_+ \mathscr{Q}_1 \bar{\phi})\, dx\, dt \\
&= -\lim_{n\to\infty} \int_0^\infty \left[\int_0^\infty \{\partial_t(\omega_n \sigma_+ \mathscr{Q}_1\bar{\phi})\} - (\partial_t\omega_n)\,\sigma_+ \mathscr{Q}_1 \bar{\phi}\, dt\right] dx \\
&= \lim_{n\to\infty} \int_0^\infty \int_0^\infty (\partial_t \omega_n)\,\sigma_+ \mathscr{Q}_1 \bar{\phi}\, dx\, dt \quad \text{since} \quad (\mathscr{Q}_1\bar{\phi})(x,0) \equiv 0 \\
&\qquad\qquad \text{because} \quad \bar{\phi}(x,0) \equiv 0 \\
&= \lim_{n\to\infty} \langle\langle \partial_t\omega_n, \sigma_+ \mathscr{Q}_1\phi\rangle\rangle = \langle\langle \partial_t v, \sigma_+ \mathscr{Q}_1 \phi\rangle\rangle \\
&= \langle\langle \sigma_+ \partial_t v, \mathscr{Q}_1\phi\rangle\rangle = \langle\langle \mathscr{Q}(\sigma_+ \partial_t v), \phi\rangle\rangle
\end{aligned}$$

where $\mathscr{Q} = \mathscr{Q}_1^*: L^2(\mathbb{R}_+; H^{-1}(\mathbb{R}_+)) \to \mathscr{K}_{-1}^+$ is the adjoint of $\mathscr{Q}_1$. Thus

$$|\sigma_1|\, \partial_t(\mathscr{P}v) = \mathscr{Q}(\sigma_+ \partial_t v) \in \mathscr{K}_{-1}^+ \tag{8.15}$$

and

$$|||\, |\sigma_1|\, \partial_t(\mathscr{P}v)|||_{-1} \leqslant \|\mathscr{Q}\|\, |||\sigma_+ \partial_t v|||_{-1}.$$

Thus $|\sigma_0|\, \partial_t(\mathscr{P}u) \in \mathscr{K}_{-1}$ for all $u \in K$ and $\mathscr{P}: K \to W$ is continuous. This completes the proof of the Extension theorem. ∎

We are finally in a position to complete the proof of the Trace theorem. Recall that all that had remained to be proven was the validity of inequality (7.19). So let $v_1 \in K$. Then

$$\int_0^{\infty} |\sigma_+(x, 0)|\,|v_1(x, 0)|^2\,dx = \int_0^{\infty} |\sigma_+(x, 0)|\,|(\mathscr{P}v_1)(x, 0)|^2\,dx$$

$$\leqslant \int_{-\infty}^{\infty} |\sigma_1(x, 0)|\,|(\mathscr{P}v_1)(x, 0)|^2\,dx \leqslant C_0^2\,\|v_1\|_K^2,$$

by the Extension theorem.

The proof is now done. ▮ ▮ ▮

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